PERTH MODERN SCHOOL
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Independent Public School

Course Specialist Year 11 Test 2 2022

Student name: ________________ Teacher name: ________________

Task type: **Response**

Time allowed for this task: 40 mins

Number of questions: 5

Materials required: Calculator with CAS capability (to be provided by the student)

Standard items: Pens (blue/black preferred), pencils (including coloured), sharpener, correction fluid/tape, eraser, ruler, highlighters

Special items: Drawing instruments, templates, notes on one unfolded sheet of A4 paper, and up to three calculators approved for use in the WACE examinations

Marks available: **40 marks**

Task weighting: **10 %**

Formula sheet provided: Yes

Note: All part questions worth more than 2 marks require working to obtain full marks.

Question 1 **(6 marks)**

Consider the three vectors $\boldsymbol{a} = 2\boldsymbol{i} - 3\boldsymbol{j}$, $\boldsymbol{b} = 3\boldsymbol{i} + m\boldsymbol{j}$ and $\boldsymbol{c} = \boldsymbol{i} - \boldsymbol{2j}$, where $m \in \mathbb{R}$.

(a) Find the value(s) of m for which $|b| = 5\sqrt{3}$. (2 marks)

Solution
$|b| = \sqrt{9 + m^2} = 5\sqrt{3}$ $9 + m^2 = 75$ $m^2 = 75 - 9 = 66$ $m = \pm\sqrt{66}$
Specific behaviours
✓ equation from magnitude ✓ both values of m

(b) Calculate $\boldsymbol{a} - \boldsymbol{2b}$, leave your answer in terms of m. (2 marks)

Solution
$\boldsymbol{a} - \boldsymbol{2b} = \begin{pmatrix} 2 \\ -3 \end{pmatrix} - 2\begin{pmatrix} 3 \\ m \end{pmatrix} = \begin{pmatrix} -4 \\ -3 - 2m \end{pmatrix}$
Specific behaviours
✓ correct multiple ✓ correct vector

(c) Find the value(s) of m such that $\boldsymbol{a} - \boldsymbol{2b}$ is perpendicular to $\boldsymbol{c}$. (2 marks)

Solution
$(\boldsymbol{a} - \boldsymbol{2b}) \cdot \boldsymbol{c} = \boldsymbol{0}$ $\begin{pmatrix} -4 \\ -3 - 2m \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -2 \end{pmatrix} = -4 + 6 + 4m = 0$ $4m = -2$ $m = -\frac{1}{2}$
Specific behaviours
✓ equation from perpendicular ✓ correct value for m

Question 2 **(9 marks)**

(a) Determine the number of integers between 1 and 450 inclusive that are divisible by 2 or 7. (4 marks)

Solution
$450 \div 2 = 225 \Rightarrow 225 \text{ divisible by } 2$ $450 \div 7 = 64.28\ldots \Rightarrow 64 \text{ divisible by } 7$ $450 \div 14 = 32.14\ldots \Rightarrow 32 \text{ divisible by both}$ $n = 225 + 64 - 32 = 257$
Specific behaviours
✓ divisible by 2 & 7 ✓ divisible by 14 ✓ use of inclusion-exclusion principle ✓ correct number

(b) A selection of three athletes is to be formed from 5 Australian, 7 American and 6 European athletes. Determine the number of different selections of three athletes if

(i) there are no restrictions. (1 mark)

Solution
$\binom{18}{3} = 816$
Specific behaviours
✓ correct number

(ii) the selection must have one athlete from each continent. (2 marks)

Solution
$\binom{5}{1} \times \binom{7}{1} \times \binom{6}{1} = 210$
Specific behaviours
✓ uses multiplication principle ✓ correct number

(iii) the selection must have at least two athletes from Australia. (2 marks)

Solution
$\binom{5}{2}\binom{13}{1} + \binom{5}{3}\binom{13}{0} = 130 + 10 = 140$
Specific behaviours
✓ indicates two cases ✓ correct number

Question 3 **(8 marks)**

Relative to the origin O, Points A and B have position vectors $\boldsymbol{a} = 4\boldsymbol{i} - 3\boldsymbol{j}$ and $\boldsymbol{b} = -2\boldsymbol{i} + \boldsymbol{j}$, respectively.

a) Determine the exact unit vector $\hat{\mathbf{c}}$, where $\mathbf{c} = \overrightarrow{AB}$. (3 marks)

Solution
$\mathbf{c} = \begin{pmatrix} -2 \\ 1 \end{pmatrix} - \begin{pmatrix} 4 \\ -3 \end{pmatrix} = \begin{pmatrix} -6 \\ 4 \end{pmatrix}$ $\lvert\mathbf{c}\rvert = \sqrt{36 + 16} = 2\sqrt{13}$ $\hat{\mathbf{c}} = \frac{1}{2\sqrt{13}}\begin{pmatrix} -6 \\ 4 \end{pmatrix} = \frac{\sqrt{13}}{13}\begin{pmatrix} -3 \\ 2 \end{pmatrix}$
Specific behaviours
✓ vector $\mathbf{c}$ ✓ magnitude   ✓ unit vector

b) Vector $\mathbf{d}$ has magnitude $5\sqrt{13}$, is parallel to $\mathbf{c}$ and in the opposite direction. Determine $\mathbf{d}$. (2 marks)

Solution
$\mathbf{d} = 5\sqrt{13} \times (-1) \times \frac{\sqrt{13}}{13}\begin{pmatrix} -3 \\ 2 \end{pmatrix}$ $= \begin{pmatrix} 15 \\ -10 \end{pmatrix}$
Specific behaviours
✓ reverses $\mathbf{c}$ ✓ correct vector

c) Determine the vector projection of $\boldsymbol{a}$ in the direction of $\boldsymbol{b}$. (3 marks)

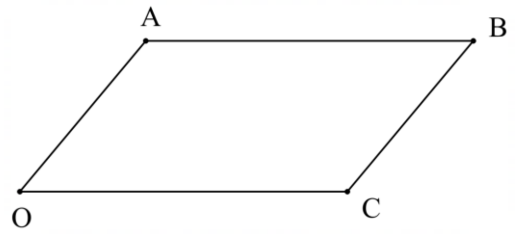
Solution
$\hat{\boldsymbol{b}} = \frac{1}{\sqrt{5}}\begin{pmatrix} -2 \\ 1 \end{pmatrix}$ $\boldsymbol{a}.\hat{\boldsymbol{b}} = \begin{pmatrix} 4 \\ -3 \end{pmatrix}.\frac{1}{\sqrt{5}}\begin{pmatrix} -2 \\ 1 \end{pmatrix} = \frac{-11}{\sqrt{5}}$ Hence required vector is $\boldsymbol{a}.\hat{\boldsymbol{b}} \times \hat{\boldsymbol{b}} = \frac{-11}{\sqrt{5}} \times \frac{1}{\sqrt{5}}\begin{pmatrix} -2 \\ 1 \end{pmatrix} = \begin{pmatrix} 22/5 \\ -11/5 \end{pmatrix}$
Specific behaviours
✓ unit vector for $\mathbf{b}$ ✓ scalar product   ✓ correct vector

Alternative Solution
$\boldsymbol{a} = 4\boldsymbol{i} - 3\boldsymbol{j}$ and $\boldsymbol{b} = -2\boldsymbol{i} + \boldsymbol{j}$ $$\boldsymbol{a}.\boldsymbol{b} = \begin{pmatrix} \mathbf{4} \\ \mathbf{-3} \end{pmatrix} . \begin{pmatrix} \mathbf{-2} \\ \mathbf{1} \end{pmatrix} = \mathbf{-8 - 3 = -11}$$ $$\boldsymbol{b}.\boldsymbol{b} = \begin{pmatrix} \mathbf{-2} \\ \mathbf{1} \end{pmatrix} . \begin{pmatrix} \mathbf{-2} \\ \mathbf{1} \end{pmatrix} = \mathbf{4 + 1 = 5}$$ Hence required vector is $$\frac{\boldsymbol{a}.\boldsymbol{b}}{\boldsymbol{b}.\boldsymbol{b}} \times \boldsymbol{b} = \frac{-11}{5} \times \begin{pmatrix} -2 \\ 1 \end{pmatrix} = \begin{pmatrix} {}^{22}/_{5} \\ -{}^{11}/_{5} \end{pmatrix}$$

Specific behaviours
✓ scalar product $\boldsymbol{a}.\boldsymbol{b}$ ✓ scalar product $\boldsymbol{b}.\boldsymbol{b}$ ✓ correct vector

Question 4 **(8 marks)**

Let $OABC$ be a parallelogram, with sides $\overrightarrow{OA} = \mathbf{a}, \overrightarrow{OC} = \mathbf{c}$



(a) Determine the vectors $\overrightarrow{OB}$ and $\overrightarrow{AC}$ in terms of $\mathbf{a}$ and $\mathbf{c}$. (2 marks)

Solution
$\overrightarrow{OB} = \boldsymbol{a} + \boldsymbol{c}$ $\overrightarrow{AC} = \boldsymbol{c} - \boldsymbol{a}$
Specific behaviours
✓ correct expression $\overrightarrow{OB}$ ✓ correct expression $\overrightarrow{AC}$.

(b) Use a vector method, to prove that the sum of the squares of the lengths of the diagonals of $OABC$, is equal to the sum of the squares of the lengths of all four sides of $OABC$. (3 marks)

Solution
$\lvert\boldsymbol{a}+\boldsymbol{c}\rvert^2 + \lvert\boldsymbol{c}-\boldsymbol{a}\rvert^2 = (\boldsymbol{a}+\boldsymbol{c}).(\boldsymbol{a}+\boldsymbol{c}) + (\boldsymbol{c}-\boldsymbol{a}).(\boldsymbol{c}-\boldsymbol{a})$ $= \boldsymbol{a}.\boldsymbol{a} + \boldsymbol{a}.\boldsymbol{c} + \boldsymbol{c}.\boldsymbol{a} + \boldsymbol{c}.\boldsymbol{c} + \boldsymbol{c}.\boldsymbol{c} - \boldsymbol{a}.\boldsymbol{c} - \boldsymbol{c}.\boldsymbol{a} + \boldsymbol{a}.\boldsymbol{a}$ $= \lvert\boldsymbol{a}\rvert^2 + \lvert\boldsymbol{c}\rvert^2 + \lvert\boldsymbol{a}\rvert^2 + \lvert\boldsymbol{c}\rvert^2$ $= 2(\lvert\boldsymbol{a}\rvert^2 + \lvert\boldsymbol{c}\rvert^2)$ ∴ The sum of the squares of the lengths of the diagonals of $OABC$, is equal to the the sum of the squares of the lengths of all four sides of $OABC$
Specific behaviours
✓ Rewrite LHS using dot product ✓ $Expands$ the expression ✓ Simplify to prove RHS

(b) What can be said about the parallelogram if the diagonals are perpendicular? (3 marks)

Solution
$If\ the\ diagonals\ are\ perpendicular\ then\ \overrightarrow{OB} \perp \overrightarrow{AC} = 0.$ Thus $(\boldsymbol{a}+\boldsymbol{c}).(\boldsymbol{c}-\boldsymbol{a}) = \mathbf{0}$ Which implies that $\lvert\boldsymbol{c}\rvert^2 = \lvert\boldsymbol{a}\rvert^2 \Longrightarrow \lvert\boldsymbol{a}\rvert = \lvert\boldsymbol{c}\rvert$ Since all sides are equal, the parallelogram is a rhombus.
Specific behaviours
✓ Considers the dot product of the two diagonals ✓ Deduces that the sides have same length ✓ Makes a concluding statement that parallelogram is a rhombus

Question 5 **(9 marks)**

a) A dry cleaner selects shirts at random from a laundry bag. The laundry bag is known to contain 2 white shirts, 3 green shirts, 5 blue shirts, 8 yellow shirts and 10 red shirts.

Use the pigeonhole principle to determine how many shirts have to be drawn to guarantee that the dry cleaner has at least three of one colour. Justify your answer. (3 marks)

Solution
Among green, blue, yellow, and red which has more than 3 shirts each, we need to pick $4\times 2+1=9$. We can get all the white shirts before the above 9, therefore we need to add 2 white as well. Hence, $2+9=11$
Specific Behaviour
✓ Uses pigeonhole principle to conclude 9 needed from green, blue, yellow, and red ✓ Adds 2 white to the total ✓ Determines the correct total

b) In Western Australia all motorcycle licence plates begin with a 1, followed by three letters and then three numbers, to create the 7-character registration plate standard.
Example: 1AAA.000

(i) How many different licence plates are possible? (2 marks)

Solution
Number of possible number plates $= 1\times 26^3\times 10^3$ $= 17576000$
Specific behaviours
✓ correct expression ✓ correct answer

(ii) In March 2020, the general plates had reached the "1H" sequence. How many different licence plates are possible, with this sequence? (1 mark)

Solution
Number of possible number plates $= 1\times 26\times 26\times 10^3$ $= 676000$
Specific behaviours
✓ correct answer

(iii) How many different plates contain only odd digits and exactly two vowels, with no repeated letters or digits? (3 marks)

Solution
Number of possible number plates $^{4}\mathbf{P}_3 \times {}^{5}\mathbf{C}_2 \times {}^{21}\mathbf{C}_1 \times 3! = 30240$
Specific behaviours
✓ indicates method/expression for choosing vowels ✓ indicates method/expression for odd digits ✓ correct answer

Extra Working Space

Question___________